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Graphical Solutions & Linearization

Unit: Laboratory & Measurement

NGSS Standards/MA Curriculum Frameworks (2016): SP4, SP5

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.B, 2.A, 2.B, 2.D, 3.C

Mastery Objective(s): (Students will be able to…)

- Use a graph to calculate the relationship between two variables.

Success Criteria:

- Graph has the manipulated variable on the *x*-axis and the responding variable on the *y*-axis.
- Graph includes best-fit line that appears to minimize the total accumulated distance between the points and the line.
- Axes and best-fit line drawn with straightedge.
- Divisions on axes are evenly spaced.
- Slope of line determined correctly (rise/run).
- Slope used correctly in calculation of desired result.

Language Objectives:

- Explain why a best-fit line gives a better answer than calculating an average.
- Explain how the slope of the line relates to the desired quantity.

Tier 2 Vocabulary: plot, axes

Notes:

Most experiments in a high-school physics class involve finding a mathematical relationship between two quantities. While it is possible to simply measure each quantity once and calculate, or to measure several times and calculate the average, an approach that measures the relationship across a range of values will provide a better result.

A common way to achieve this is to manipulate equations and plot data such that the expected result is a straight line, and the experimental quantity is calculated by plotting a best-fit line and determining the slope. This means that you need to plot graphs ***accurately***, either on graph paper or using a computer or calculator. If you use graph paper:

- The data points need to be as close to their actual locations as you are capable of drawing.
- The best-fit line needs to be as close as you can practically get to its mathematically correct location.
- The best-fit line must be drawn with a straightedge.
- The slope needs to be calculated using the actual rise and run of points on the best-fit line.

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Once you have your data points, arrange the equation into $y = mx + b$ form, such that the slope is the quantity of interest. Then accurately plot your data and draw a best-fit line. The slope of this line will be the quantity of interest.

For example, suppose you wanted to calculate the spring constant of a spring by stretching it and measuring the resulting force applied by the spring. (This will be covered in the *Springs* topic, starting on page 322.) You obtain the following data:

Displacement (m)	0	0.05	0.10	0.15	0.20	0.25	0.30
Spring Force (N)	0	0.9	1.7	2.7	4.1	5.1	5.8

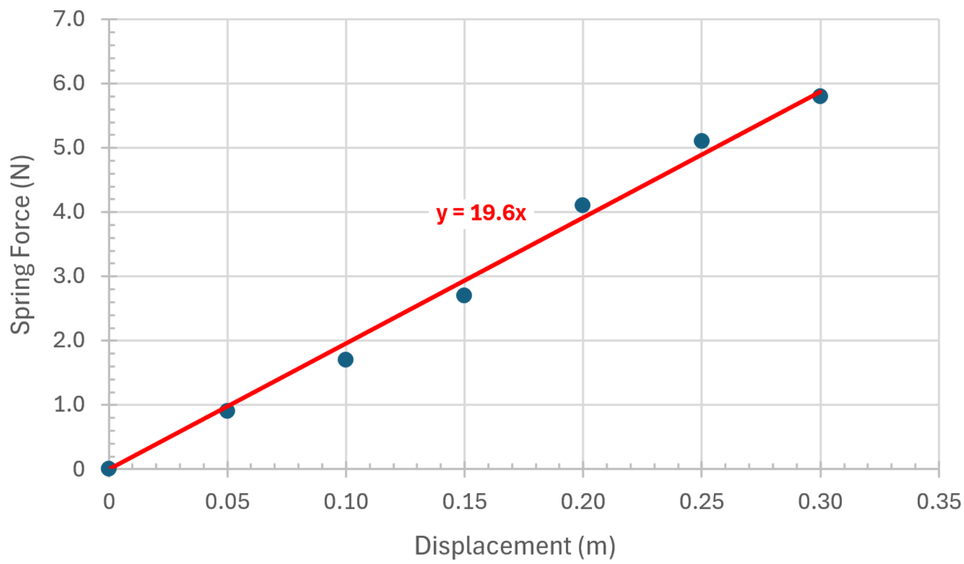
The relevant equation is Hooke's Law, $F_s = kx$. Note that Hooke's Law is already in *y* = *mx* + *b* form:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ F_s & = & k & x & + & 0 \end{array}$$

In our equation:

- F_s corresponds to *y*, so we will plot F_s (force) on the *y*-axis.
- *x* corresponds to *x*, so we will plot *x* (displacement) on the *x*-axis. ☺
- *k* corresponds to *m* (the slope), so the slope of our graph will be the spring constant *k*. (Recall that this is the quantity that we want.)

The plot looks like the following:



Conveniently, the spreadsheet that was used to plot the best-fit line (trendline) is able to display the equation for the line. The slope is 19.6, which means our spring constant is $19.6\,\frac{\text{N}}{\text{m}}$.

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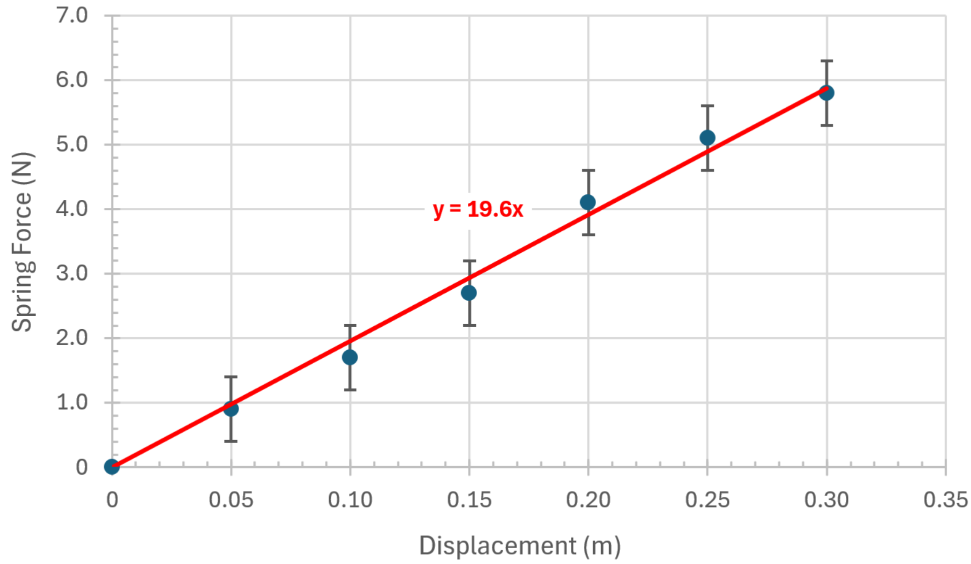
Uncertainty and Best-Fit lines

If you determine a quantity by linearizing, the best-fit line should pass within the uncertainty of every data point. (This is, after all, what uncertainty means!) You can show/check this by plotting the graph with error bars, which show the maximum and minimum values for each data point, based on the uncertainty.

Suppose our force measurements each had an uncertainty of ± 0.5 N. The best-fit line with error bars would look like this:



The top of each error bar is the force plus the uncertainty, which is the maximum possible value (assuming we have estimated our uncertainty appropriately). The bottom of each error bar is the force minus the uncertainty, which is the minimum possible value.

Notice that the best fit line passes through all of the error bars. This is important. ***If the best-fit line calculated by linear regression does not pass through the error bars, the equation of the line must be manipulated until it does.*** (This is, after all, what uncertainty means!)

If it is not possible to plot a line that passes through all of the error bars, this suggests that either:

- The uncertainty was underestimated.
- The data points whose error bars do not intersect the line may be outliers.

If a data point is an outlier, you should attempt to determine the cause—it was most likely an unidentified problem with that data point. (If there is a problem with your results, the first thing you should check is your procedure.)

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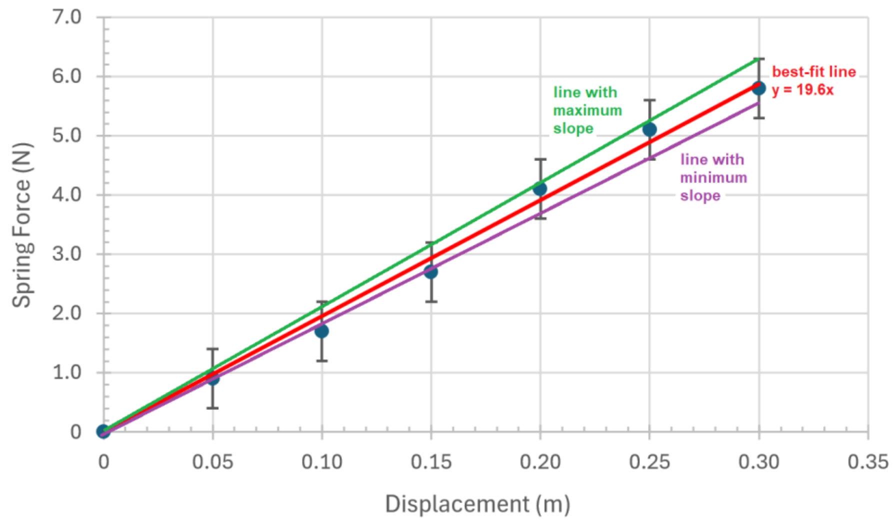
Calculating the uncertainty of a value that was calculated from a best-fit line is difficult. The largest possible value is the line with maximum slope that passes through all of the error bars (shown in green below). The smallest possible value is the line with minimum slope that passes through all of the error bars (shown in purple).



(Note that because zero force must result in zero displacement, the intercept of all of the lines was forced through zero.)

Manipulating the equations of the two "worst-fit" lines is tedious, and beyond the scope of a high school course.

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Linearization

Often, it is desirable to use linear regression (the process of calculating the best-fit line) in situations where the equation itself is non-linear. For example, suppose you want to determine the electrical current (I) passing through a circuit. You know the total electrical resistance of the circuit (R), and you are able to measure the power consumption (P).

The equation relating these quantities is $P = I^2 R$. In slope-intercept form, this looks like:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ P & = & I^2 & R & + & 0 \end{array}$$

This means you should plot a graph of P vs. R. You should force the intercept of the best-fit line through zero, and the slope will be I^2. Once you determine the slope, you need to take the square root of it to get the value of I.

Suppose instead that you had the same electrical circuit, but you were able to measure power (P) and current (I), and you wanted to determine the resistance (R). The equation is still $P = I^2 R$, which we can rewrite as $P = R I^2$. Now, the slope-intercept form of our equation is:

$$\begin{array}{ccccccc} y & = & m & x & + & b \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ P & = & R & I^2 & + & 0 \end{array}$$

This time, we need to plot a graph of P vs. I^2. Again, you should force the best-fit line through zero, and the slope will be R.

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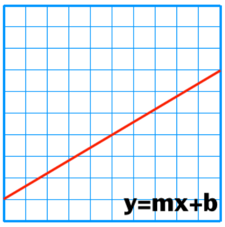
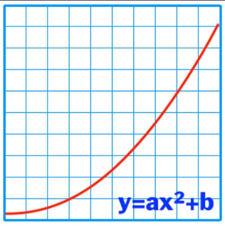
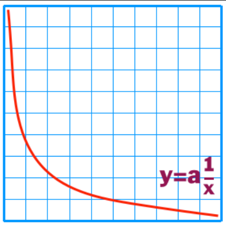
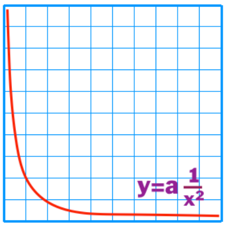
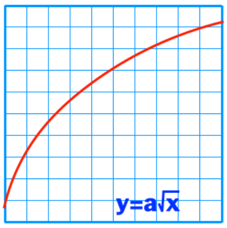
Recognizing Shapes of Graphs

When you know the equation in advance, it is easy to rearrange the equation in order to linearize it. However, if you do not know the equation before looking at the data, you can often make a guess based on its shape.

It is useful to memorize the general shapes of these graphs* so you can predict the type of relationship and the form of the equation. Note, however, that some graphs look similar to others. Just because a graph "looks like" it fits a particular equation doesn't necessarily mean that the equation is correct!

Plot of *y vs. x*	Equation	Linear Plot
	**Linear** $y = mx + b$ b = y-intercept	y vs. x slope = m
	**Power** $y = ax^2$ or $y = ax^2 + b$ b = minimum y-value	y vs. x^2 slope = a
	**Inverse** $y = \frac{a}{x}$ or $y = a \cdot \frac{1}{x}$ undefined (∞) at $x = 0$	y vs. $\frac{1}{x}$ slope = a
	**Inverse Square** $y = \frac{a}{x^2}$ or $y = a \cdot \frac{1}{x^2}$ undefined (∞) at $x = 0$	y vs. $\frac{1}{x^2}$ slope = a
	**Square Root** $y = a\sqrt{x}$	y vs. $\sqrt{x}$ slope = a

*Graphs by Tony Wayne. Used with permission.

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