Equations of Motion

Unit: Kinematics (Motion) in One Dimension

NGSS Standards/MA Curriculum Frameworks (2016): HS-PS2-10(MA)

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.3.A.1, 1.3.A.2, 1.3.A.3

Mastery Objective(s): (Students will be able to…)

- Use the equations of motion to calculate position, velocity and acceleration for problems that involve motion in one dimension.

Success Criteria:

- Vector quantities position, velocity, and acceleration are identified and substituted correctly, including sign (direction).
- Time (scalar) is correct and positive.
- Algebra is correct and rounding to appropriate number of significant figures is reasonable.

Language Objectives:

- Correctly identify quantities and assign variables in word problems.

Tier 2 Vocabulary: position, displacement, velocity, acceleration, direction

Notes:

As previously noted, average velocity is the displacement (change in position) with respect to time. (*E.g.,* if your displacement is 10 m over a period of 2 s, then your average velocity is $\vec{v}_{ave.} = \frac{\vec{d}}{t} = \frac{10}{2} = 5\,\frac{\text{m}}{\text{s}}$.)

Derivations of Equations

We can rearrange the formula for average velocity to show that displacement is average velocity times time:

$$\vec{v}_{ave.}(t) = \frac{\vec{d}}{\cancel{t}}(\cancel{t}) \quad \rightarrow \quad \vec{d} = (\vec{v}_{ave.})(t)$$

Note that when an object's velocity is changing, the initial velocity $\vec{v}_o$, the final velocity, $\vec{v}$, and the average velocity, $\vec{v}_{ave.}$ are *different quantities* with *different values*. (This is a common mistake that first-year physics students make.) Assuming acceleration is constant*, the average velocity is just the average of the initial and final velocities. This gives the following equation:

$$\vec{v}_{ave.} = \frac{\vec{v}_o + \vec{v}}{2} = \frac{\vec{d}}{t}$$

* In an algebra-based physics course, we will limit ourselves to problems in which acceleration is constant.

Use this space for summary and/or additional notes:

Acceleration is a change in velocity over a period of time. This means that formula for acceleration is:

$$\vec{a}_{ave.} = \frac{\vec{v} - \vec{v}_o}{t} = \frac{\Delta\vec{v}}{t} = \frac{\Delta\vec{v}}{\Delta t}$$

We can rearrange this formula to show that the change in velocity is acceleration times time:

$$\Delta\vec{v} = \vec{v} - \vec{v}_o = \vec{a}t$$

We can combine the formula for average velocity with the formula for acceleration in order to get a formula for the position of an object that is accelerating.

$$\boldsymbol{d} = (\boldsymbol{v}_{ave.})(t)$$
$$\boldsymbol{v} = \boldsymbol{a}t$$

However, the problem is that $\boldsymbol{v}$ in the formula $\boldsymbol{v} = \boldsymbol{a}t$ is the velocity at the *end*, which is not the same as the *average* velocity $\boldsymbol{v}_{ave.}$.

If the velocity of an object is changing at a constant rate (*i.e.,* the object is accelerating uniformly), the average velocity, $\boldsymbol{v}_{ave.}$ is given by the formula:

$$\boldsymbol{v}_{ave.} = \frac{\boldsymbol{v}_o + \boldsymbol{v}}{2}$$

To make the math easier to follow, let's start by assuming that the object starts at rest (not moving, which means $\boldsymbol{v}_o = 0$) and it accelerates at a constant rate. The average velocity is therefore the average of the initial velocity and the final velocity:

$$\boldsymbol{v}_{ave.} = \frac{\overset{0}{\cancel{\boldsymbol{v}_o}} + \boldsymbol{v}}{2} = \frac{0 + \boldsymbol{v}}{2} = \frac{\boldsymbol{v}}{2} = \frac{1}{2}\boldsymbol{v}$$

Combining all of these gives the following, for an object starting from rest:

$$\boldsymbol{d} = \boldsymbol{v}_{ave.}t = \tfrac{1}{2}\boldsymbol{v}t \quad \rightarrow \quad \vec{d} = \tfrac{1}{2}\boldsymbol{v}t = \tfrac{1}{2}(\boldsymbol{a}t)\,t = \tfrac{1}{2}\boldsymbol{a}t^2$$

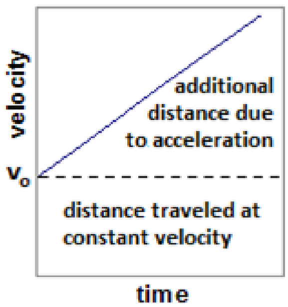
Now, recall from above that $\vec{d} = \vec{v}_{ave.}t$. Suppose that instead of starting from rest, an object's velocity is constant. The initial velocity is therefore also the final velocity and the average velocity, $(\vec{v}_o = \vec{v} = \vec{v}_{ave.})$, which means at constant velocity $\vec{d} = \vec{v}_o t$.

Therefore, if the object does not start from rest and it accelerates, we can combine these two formulas, resulting in:

$$\vec{d} = \vec{v}_o t + \tfrac{1}{2}\vec{a}t^2$$

distance the object would travel if its initial velocity were constant

additional distance the object will travel because it is accelerating



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Finally, we can combine the equation $\vec{d} = \vec{v}_o t + \frac{1}{2}\vec{a}t^2$ with the equation $\vec{v} - \vec{v}_o = \vec{a}t$ and eliminate time, giving the following equation, which relates initial and final velocity and distance:

$$\vec{v}^2 - \vec{v}_o^2 = 2\vec{a}\vec{d}^{\,*}$$

The algebra is straightforward but tedious, and will not be presented here.

Summary of Motion Equations

Most motion problems can be calculated from Isaac Newton's equations of motion. The following is a summary of the equations presented in the previous sections:

Equation	Variables					Description
$\frac{\vec{d}}{t} = \frac{\vec{v}_o + \vec{v}}{2} \left(= \vec{v}_{ave.}\right)$	$\vec{d}$	$\vec{v}_o$	$\vec{v}$		t	Average velocity is the distance per unit of time, which also equals the calculated value of average velocity.
$\vec{v} - \vec{v}_o = \vec{a}t$		$\vec{v}_o$	$\vec{v}$	$\vec{a}$	t	Acceleration is a change in velocity divided by time.
$\vec{d} = \vec{v}_o t + \frac{1}{2}\vec{a}t^2$	$\vec{d}$	$\vec{v}_o$		$\vec{a}$	t	Total displacement is the displacement due to velocity ($\vec{v}_o t$), plus the displacement due to acceleration ($\frac{1}{2}\vec{a}t^2$).
$\vec{v}^2 - \vec{v}_o^2 = 2\vec{a}\vec{d}$	$\vec{d}$	$\vec{v}_o$	$\vec{v}$	$\vec{a}$		Velocity at the end can be calculated from velocity at the beginning, acceleration, and displacement.

* Note that this is not a proper vector expression. Vector multiplication is either a dot product, a cross product, or a tensor product; the expressions $\vec{v}^2$ and $\vec{a}\vec{d}$ are meaningless as vector expressions. The equation is presented this way to remind students that $\vec{v}$, $\vec{v}_o$, $\vec{a}$, and $\vec{d}$ are each vectors, whose signs (in one dimension) are positive or negative depending on direction.

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Representing Vectors with Positive and Negative Numbers

Remember that position ($\vec{x}$), velocity ($\vec{v}$), and acceleration ($\vec{a}$) are all vectors, which means each of them can be positive or negative, depending on the direction.

- If an object is located is on the positive side of the origin (position zero), then its position, $\vec{x}$, is positive. If the object is located on the negative side of the origin, its position is negative.
- If an object is moving in the positive direction, then its velocity, $\vec{v}$, is positive. If the object is moving in the negative direction, then its velocity is negative.
- If an object's velocity is "trending positive" (increasing in the positive direction *or decreasing in the negative direction*), then its acceleration, $\vec{a}$, is positive. If the object's velocity is "trending negative" (decreasing in the positive direction *or increasing in the negative direction*), then its acceleration is negative.
- An object can have positive velocity and negative acceleration at the same time (or *vice versa*).
- An object can have a velocity of zero (for an instant) but can still be accelerating.

Selecting the Appropriate Equation

When you are faced with a problem, choose an equation based on the following criteria:

- The equation *must* contain the variable you are looking for.
- All other quantities in the equation must be either given in the problem or assumed from the description of the problem.
  - If an object *starts from rest* (not moving), that means $\vec{v}_o = 0$.
  - If an object *comes to rest* (*stops*), that means $\vec{v} = 0$. (Remember that $\vec{v}$ is the velocity at the end.)
  - If an object is moving at a constant velocity, then $\vec{v} = \text{constant} = \vec{v}_o = \vec{v}_{ave.}$ and $\vec{a} = 0$.
  - If the object is in free fall*, that means $\vec{a} = \vec{g} \approx 10\,\frac{\text{m}}{\text{s}^2}$ downward. Look for words like *drop*, *fall*, *throw*, *etc.* (Does not apply to rotation problems.)

This means you can choose the appropriate equation by making a list of what you are looking for and what you know. The equation in which you know everything except what you are looking for is the one to use.

* See below.

Use this space for summary and/or additional notes:

Free Fall (Acceleration Caused by Gravity)

The gravitational force (or "force of gravity") is an attraction between objects that have mass.

free fall: when an object is freely accelerating toward the center of the Earth (or some other object with a very large mass) because of the effects of gravity, and the effects of other forces are negligible.

Objects in free fall on Earth accelerate downward at a rate of approximately $10\,\frac{\text{m}}{\text{s}^2} \approx 32\,\frac{\text{ft.}}{\text{s}^2}$. (The actual value is approximately $9.806\,\frac{\text{m}}{\text{s}^2}$ at sea level near the surface of the Earth. In this course we will usually round it to $10\,\frac{\text{m}}{\text{s}^2}$ so the calculations don't get in the way of understanding the physics.) When an object is in free fall, we usually replace the variable $\vec{\boldsymbol{a}}$ with the constant $\vec{\boldsymbol{g}}$.

Note that an object going down a ramp is ***not*** in free fall, even though gravity is the force that caused the object to accelerate. The object's motion is constrained by the ramp and it is not free to fall straight down.

As with any other vector quantity, acceleration due to gravity can be represented by a positive or negative number, depending on which direction you choose to be positive. For example, if we choose "up" to be the positive direction, that would mean acceleration due to gravity is in the negative direction, *i.e.,* $\vec{\boldsymbol{a}} = \vec{\boldsymbol{g}} = -10\,\frac{\text{m}}{\text{s}^2}$.

Hints for Solving Problems Involving Free Fall

1. If an object is thrown upwards, gravity will cause it to accelerate downwards. This means that if we choose the positive direction to be "up," $\vec{\boldsymbol{v}}_o$ will be positive, but $\vec{\boldsymbol{a}}$ will be $-10\,\frac{\text{m}}{\text{s}^2}$ (*i.e.,* negative because it's downwards).
2. At an object's *maximum height*, it stops moving for an instant ($\vec{\boldsymbol{v}} = 0$).
3. If an object goes up and then falls down to the *same height* it started from:
    a. There is no (vertical) displacement $(\vec{\boldsymbol{d}} = 0)$.
    b. *The time that the object spends going upwards is the same as the time it spends going downwards.* The time it takes to reach its maximum height is therefore half of the total time it takes to go up to its highest point and return to the ground.
    c. The *magnitude* of the velocity at the end will be the same as at the beginning, but the direction will be opposite. ($\vec{\boldsymbol{v}} = -\vec{\boldsymbol{v}}_o$)

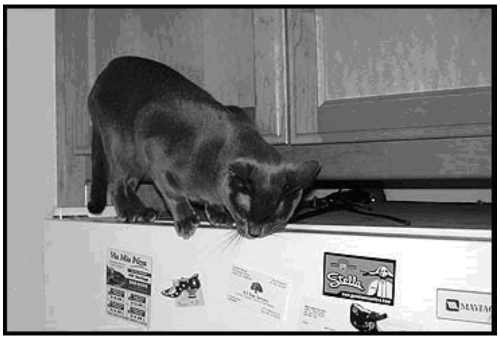
Use this space for summary and/or additional notes:

A Strategic Approach to Problem Solving

When solving motion problems, it can help to make a table of values and directions to keep track of each quantity.

Sample problems:

Q: If a cat jumps off a 1.8 m tall refrigerator, how fast is it going just before it hits the ground?



A: The cat is starting from rest ($\vec{v}_o = 0$), and acceleration due to gravity is $\vec{a} = \vec{g} = 10\,\frac{\text{m}}{\text{s}^2}$ downwards. We need to find $\vec{v}$.

Because all of the vector quantities are in the downward direction, we will make "down" the positive direction.

var.	dir.	value
$\vec{d}$	↓	+1.8
$\vec{v}_o$	—	0
$\vec{v}$	?	?
$\vec{a}$	↓	+10
t	N/A	—

$$\frac{\vec{d}}{t} = \frac{\vec{v}_o + \vec{v}}{2}$$

$$\vec{v} - \vec{v}_o = \vec{a}t$$

$$\vec{d} = \vec{v}_o t + \tfrac{1}{2}\vec{a}t^2$$

$$\boxed{\vec{v}^2 - \vec{v}_o^{\,2} = 2\vec{a}\vec{d}}$$

Because both of the nonzero vector quantities are downward, we will make downward the positive direction.

Using the "GUESS" method, the only equation that has the Unknown ($\vec{v}$) and the Givens ($\vec{d}, \vec{v}_o$, and $\vec{a}$) is the fourth one.

$$\vec{v}^2 - \cancelto{0}{\vec{v}_o^{\,2}} = 2\vec{a}\vec{d}$$

$$\vec{v} = \pm\sqrt{2\vec{a}\vec{d}}$$

(Note that because we introduced the square root sign, we have to consider both the positive and negative result.)

$$\vec{v} = \pm\sqrt{2\vec{a}\vec{d}} = \pm\sqrt{(2)(10)(1.8)} = \pm\sqrt{36} = \pm 6\,\frac{\text{m}}{\text{s}}$$

It is obvious from the problem that the cat is moving downward just before it hits the ground. Because downward is the positive direction, this means that the final velocity is $+6\,\frac{\text{m}}{\text{s}}$.

Use this space for summary and/or additional notes:

Q: A student throws an apple upward with a velocity of $8\,\frac{\text{m}}{\text{s}}$. The apple comes back down and hits Sir Isaac Newton in the head, at the same height as the apple was thrown.

How much time elapsed between when the apple was thrown and when it hit Newton?

A: Once again, we make a table of quantities and directions:

var.	dir.	value
$\vec{d}$	—	0
$\vec{v}_o$	↑	+8
$\cancel{\vec{v}}$	—	—
$\vec{a}$	↓	−10
t	?	N/A

$$\frac{\vec{d}}{t} = \frac{\vec{v}_o + \vec{v}}{2}$$

$$\vec{v} - \vec{v}_o = \vec{a}t$$

$$\boxed{\vec{d} = \vec{v}_o t + \tfrac{1}{2}\vec{a}t^2}$$

$$\vec{v}^2 - \vec{v}_o^2 = 2\vec{a}\vec{d}$$

Note that because the apple landed at the same height as it was thrown, displacement is zero. Note also that because $\vec{v}_o$ is upward and $\vec{a}$ is downward, they need to have opposite signs. It doesn't matter which direction we choose to be positive, so for this problem let's arbitrarily choose upward to be the positive direction. This means $\vec{v}_o = +8\,\frac{\text{m}}{\text{s}}$ and $\vec{a} = -10\,\frac{\text{m}}{\text{s}^2}$.

We can now solve the problem:

$$\overset{0}{\cancel{\vec{d}}} = \vec{v}_o t + \tfrac{1}{2}\vec{a}t^2$$

$$0 = \vec{v}_o t + \tfrac{1}{2}\vec{a}t^2$$

$$0 = t(v_o + \tfrac{1}{2}at)$$

$$t = 0\,,\quad \vec{v}_o + \tfrac{1}{2}\vec{a}t = 0\vec{v}_o$$

$$t = 0\,,\quad \tfrac{1}{2}\vec{a}t = -\vec{v}_o$$

$$t = 0\,,\quad t = \frac{-2\vec{v}_o}{\vec{a}}$$

$$t = 0\,,\quad t = \frac{-2(8)}{-10} = 1.6\text{ s}$$

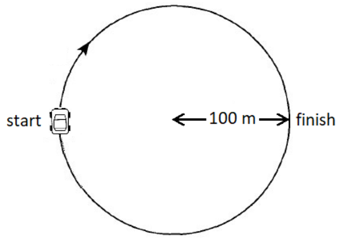
The equation helpfully tells us that the apple was at position zero twice, once at $t = 0$ when it was thrown, and again at $t = 1.6$ s when it landed on Newton's head. The problem is asking for the time when it landed, so the answer to the question that was asked is 1.6 s.

Use this space for summary and/or additional notes:

Homework Problems: Motion Equations Set #1

Try to first rearrange the equation to solve for the variable of interest and substitute numbers into the equation ***only*** after rearranging. (However, if you get stuck on a problem, feel free to solve it numerically first.)

1. **(M)** A car, traveling at constant speed, makes one lap around a circular track with a radius of 100 m. When the car has traveled halfway around the track, what *distance* did it travel? What is the magnitude of its *displacement* from the starting point?



2. **(S)** An elevator is moving upward with a speed of $11\,\frac{m}{s}$. Three seconds later, the elevator is still moving upward, but its speed has been reduced to $5.0\,\frac{m}{s}$. What is the average acceleration of the elevator during the 3.0 s interval?

   Answer: $-2\,\frac{m}{s^2}$

3. **(S – honors & AP®; M – CP1)** A car, starting from rest, accelerates in a straight-line path at a constant rate of $2.5\,\frac{m}{s^2}$. How far will the car travel in 12 seconds?

   Answer: 180 m

4. **(M – honors & AP®; S – CP1)** An object initially at rest is accelerated at a constant rate for 5.0 seconds in the positive *x* direction. If the final speed of the object is $20.0\,\frac{m}{s}$, what was the object's acceleration?

   Answer: $4\,\frac{m}{s^2}$

Use this space for summary and/or additional notes:

5. **(S – honors & AP®; M – CP1)** An object starts from rest and accelerates uniformly in a straight line in the positive *x* direction. After 10. seconds its speed is $70.\,\frac{\text{m}}{\text{s}}$.

   a. Determine the acceleration of the object.

   Answer: $7\,\frac{\text{m}}{\text{s}^2}$

   b. How far does the object travel during those first 10 seconds?

   Answer: 350 m

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6. **(M – honors & AP®; A – CP1)** A racecar has a speed of $\vec{v}_o$ when the driver releases a drag parachute. If the parachute causes a deceleration of $\vec{a}$, derive an expression for how far the car will travel before it stops.
(*If you are not sure how to do this problem, do #7 below and use the steps to guide your algebra.*)

   Answer: $\vec{d} = \frac{-\vec{v}_o^2}{2\vec{a}}$

   The negative sign means that $\vec{d}$ and $\vec{a}$ need to have opposite signs, which means they must be in opposite directions.

7. **(S)** A racecar has a speed of $80.\,\frac{\text{m}}{\text{s}}$ when the driver releases a drag parachute. If the parachute causes a *deceleration* of $4\,\frac{\text{m}}{\text{s}^2}$, how far will the car travel before it stops?
(*<u>You must start with the equations in your</u>* <u>Physics Reference Tables</u> *<u>and show all of the steps of GUESS.</u> You may only use the answer to question #6 above as a starting point if you have already solved that problem.*)

   Answer: 800 m

Use this space for summary and/or additional notes:

8. **(S – honors & AP®; A – CP1)** A ball is shot straight up from the surface of the earth with an initial speed of $30.\,\frac{\text{m}}{\text{s}}$. Neglect any effects due to air resistance.
   a. What is the maximum height that the ball will reach?

      Answer: 45 m

   b. How much time elapses between the throwing of the ball and its return to the original launch point?

      Answer: 6.0 s

9. **(S – honors & AP®; M – CP1)** A brick is dropped from rest from a height of 5.0 m. How long does it take for the brick to reach the ground?

   Answer: 1 s

Use this space for summary and/or additional notes:

10. **(M – honors & AP®; A – CP1)** A ball is dropped from rest from a tower and strikes the ground 125 m below. Approximately how many seconds does it take for the ball to strike the ground after being dropped? (Neglect air resistance.)

    Answer: 5.0 s

11. **(S – honors & AP®; M – CP1)** Water drips from rest from a leaf that is 20 meters above the ground. Neglecting air resistance, what is the speed of each water drop when it hits the ground?

    Answer: $20.0\,\frac{\text{m}}{\text{s}}$

12. **(M – honors & AP®; A – CP1)** What is the maximum height that will be reached by a stone thrown straight up with an initial speed of $35\,\frac{\text{m}}{\text{s}}$?

    Answer: 61.25 m

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Homework Problems: Motion Equations Set #2

These problems are more challenging than Set #1.

1. **(S)** A car starts from rest at 50 m to the west of a road sign. It travels to the east reaching $20\,\frac{\text{m}}{\text{s}}$ after 15 s. Determine the position of the car relative to the road sign.

   Answer: 100 m east

2. **(M)** A car starts from rest at 50 m west of a road sign. It has a velocity of $20\,\frac{\text{m}}{\text{s}}$ east when it is 50 m east of the road sign. Determine the acceleration of the car.

   Answer: $2\,\frac{\text{m}}{\text{s}^2}$

3. **(S)** During a 10 s period, a car has an average velocity of $25\,\frac{\text{m}}{\text{s}}$ and an acceleration of $2\,\frac{\text{m}}{\text{s}^2}$. Determine the initial and final velocities of the car. (*Hint: this is an algebra problem with two unknowns, so it requires two equations.*)

   Answer: $v_o = 15\,\frac{\text{m}}{\text{s}}$; $v = 35\,\frac{\text{m}}{\text{s}}$

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4. **(S)** A racing car increases its speed from an unknown initial velocity to $30\,\frac{\text{m}}{\text{s}}$ over a distance of 80 m in 4 s. Calculate the initial velocity of the car and the acceleration.

   Answer: $v_o = 10\,\frac{\text{m}}{\text{s}}$; $a = 5\,\frac{\text{m}}{\text{s}^2}$

5. **(M)** A stone is thrown vertically upward with a speed of $12.0\,\frac{\text{m}}{\text{s}}$ from the edge of a cliff that is 75.0 m high.

   a. **(M)** How much later does it reach the bottom of the cliff?

      Answer: 5.25 s

   b. **(M)** What is its velocity just before it hits the ground?

      Answer: $40.5\,\frac{\text{m}}{\text{s}}$ toward the ground ($-40.5\,\frac{\text{m}}{\text{s}}$ if "up" is positive)

   c. **(M)** What is the total distance the stone travels?

      Answer: 89.4 m

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6. **(S)** A helicopter is ascending vertically with a speed of v_o. At a height *h* above the Earth, a package is dropped from the helicopter. Derive an expression for the time, *t*, that it takes for the package to reach the ground. (*If you are not sure how to do this problem, do #7 below and use the steps to guide your algebra.*)

   Answer: $t = \dfrac{-v_o \pm \sqrt{v_o^2 - 2gh}}{g}$, disregarding the negative answer

7. **(M)** A helicopter is ascending vertically with a speed of $5.50\,\frac{\text{m}}{\text{s}}$. At a height of 100 m above the Earth, a package is dropped from the helicopter. How much time does it take for the package to reach the ground? (<u>*You must start with the equations in your* Physics Reference Tables *and show all of the steps of GUESS.*</u> *You may only use the answer to question #6 above as a starting point if you have already solved that problem.*)

   Answer: 5.06 s

8. **(S)** A tennis ball is shot vertically upwards from the ground. It takes 3.2 s for it to return to the ground. Find the total distance the ball traveled.

   Answer: 25.6 m

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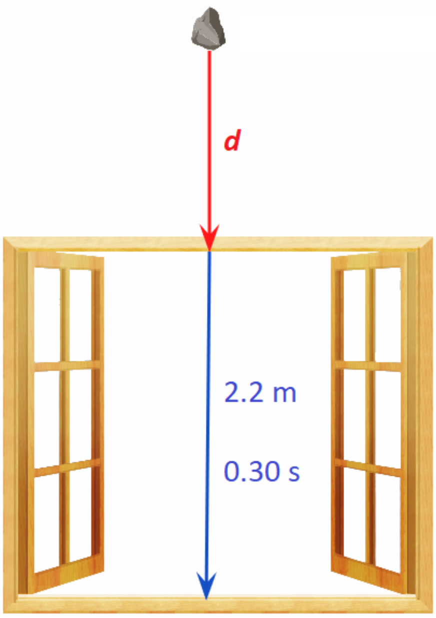
9. **(S)** A kangaroo jumps vertically to a height of 2.7 m. How long will it be in the air before returning to the earth?

Answer: 1.5 s

10. **(M –AP®; S – honors)** A falling stone takes 0.30 s to travel past a window that is 2.2 m tall. From what distance above the window, *d*, did the stone fall?



Answer: 1.70 m

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