Projectile Motion

Unit: Kinematics (Motion) in Multiple Dimensions

NGSS Standards/MA Curriculum Frameworks (2016): N/A

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.5.A, 1.5.A.1, 1.5.A.2, 1.5.A.3

Mastery Objective(s): (Students will be able to…)

- Solve problems that involve motion in two dimensions.

Success Criteria:

- Correct quantities are chosen in each dimension (x & y).
- Positive direction is chosen for each dimension and vector quantities in each dimension have the appropriate sign (+ or −).
- Time (scalar) is correct, positive, and the same in both dimensions.
- Algebra is correct and rounding to appropriate number of significant figures is reasonable.

Language Objectives:

- Correctly identify quantities with respect to type of quantity and direction in word problems.
- Assign variables correctly in word problems.

Tier 2 Vocabulary: projectile, dimension

Labs, Activities & Demonstrations:

- Play "catch."
- Drop one ball and punch the other at the same time.
- "Shoot the monkey."

Notes:

projectile: an object that is propelled (thrown, shot, *etc.*) horizontally and also falls due to gravity.

Because perpendicular vectors do not affect each other, the vertical and horizontal motion of the projectile are independent and can be considered separately, using a separate set of equations for each.

Use this space for summary and/or additional notes:

Assuming we can neglect friction and air resistance (which is usually the case in first-year physics problems), we make the following important assumptions:

Horizontal Motion

The horizontal motion of a projectile is not affected by anything except for air resistance. If air resistance is negligible, we can assume that there is no horizontal acceleration, and therefore the horizontal velocity of the projectile, $\vec{\boldsymbol{v}}_x$, is constant. This means the horizontal motion of a projectile can be described by the equation:

$$\vec{\boldsymbol{d}}_x = \vec{\boldsymbol{v}}_x t$$

The projectile is always moving in the same horizontal direction, so we make this the positive (horizontal, or "*x*") direction for the vector quantities of velocity and displacement.

Vertical Motion

Gravity affects projectiles the same way regardless of whether or not the projectile is also moving horizontally. All projectiles therefore have a constant downward acceleration of $\vec{\boldsymbol{g}} = 10\,\frac{\text{m}}{\text{s}^2}$ (in the vertical or "*y*" direction), due to gravity.

Therefore, the vertical motion of the particle can be described by the equations:

$$\vec{\boldsymbol{v}}_y - \vec{\boldsymbol{v}}_{o,y} = \vec{\boldsymbol{g}}t$$
$$\vec{\boldsymbol{d}}_y = \vec{\boldsymbol{v}}_{o,y}t + \frac{1}{2}gt^2$$
$$\vec{\boldsymbol{v}}_y^2 - \vec{\boldsymbol{v}}_{o,y}^2 = 2\vec{\boldsymbol{g}}\vec{\boldsymbol{d}}$$

(Notice that we have ***two*** subscripts for initial velocity, because it is ***both*** the initial velocity v_o ***and also*** the vertical velocity v_y.)

If the projectile is always moving downwards (*i.e.,* it is launched horizontally and it falls), we make down the positive vertical direction and all vector quantities (velocity, displacement and acceleration) in the *y*-direction are positive.

If the projectile is launched upwards, reaches a maximum height, and then falls, the velocity and displacement are sometimes upwards and sometimes downwards. In this case, we need to choose a direction to be positive. Usually, upward is chosen to be the positive direction, which makes $\vec{\boldsymbol{v}}_{o,y}$ positive, and makes $\vec{\boldsymbol{v}}_y$ and $\vec{\boldsymbol{g}}$ both negative. (In fact, $\vec{\boldsymbol{g}} = -10\,\frac{\text{m}}{\text{s}^2}$.)

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Time

The time that the projectile spends falling must be the same as the time that the projectile spends moving horizontally. This means time (t) is the same in both equations, which means time is the variable that links the vertical problem to the horizontal problem.

The consequences of these assumptions are:

- The *time* that the object takes to fall is determined by its movement *only* in the vertical direction. (When it hits the ground, it stops moving in all directions.)
- The *horizontal distance* that the object travels is determined by the time (from the vertical equation) and by its velocity in the horizontal direction.

Therefore, the general strategy for most projectile problems is:

1. Solve the vertical problem first, to get the time.
2. Use the time from the vertical problem to solve the horizontal problem.

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Sample problem:

Q: A ball is thrown horizontally at a velocity of $5\,\frac{m}{s}$ from a height of 1.5 m. How far does the ball travel (horizontally)?

A: We're looking for the horizontal distance, d_x. We know the vertical distance, $d_y = 1.5\,\text{m}$, and we know that $v_{o,y} = 0$ (there is no initial vertical velocity because the ball is thrown horizontally), and we know that $a_y = g = 10\,\frac{m}{s^2}$.

We need to separate the problem into the horizontal and vertical components.

Horizontal:

$$d_x = v_x t$$

$$d_x = 5t$$

At this point we can't get any farther, so we need to turn to the vertical problem.

Vertical:

$$d_y = v_{o,y}t + \frac{1}{2}gt^2$$

$$d_y = \frac{1}{2}gt^2$$

$$\frac{2d_y}{g} = t^2$$

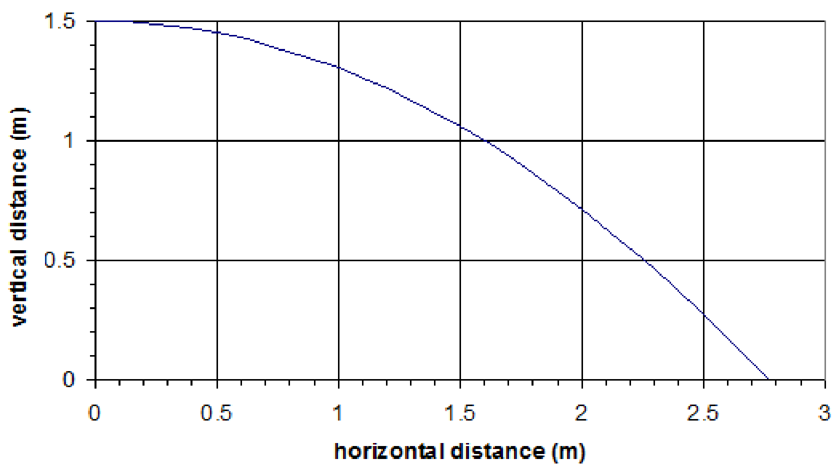
$$t = \sqrt{\frac{2d_y}{g}}$$

$$t = \sqrt{\frac{(2)(1.5)}{10}} = \sqrt{0.3} = 0.55\,\text{s}$$

Now that we know the time, we can substitute it back into the horizontal equation, giving:

$$d_x = (5)(0.55) = 2.74\,\text{m}$$

A graph of the vertical *vs.* horizontal motion of the ball looks like this:



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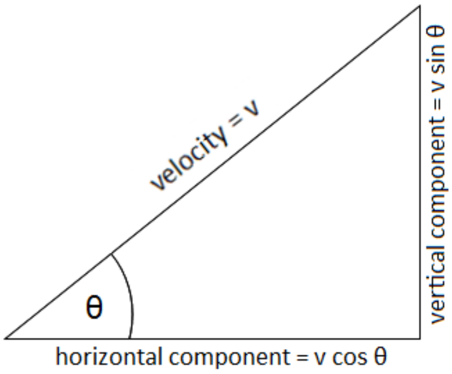
Projectiles Launched at an Angle

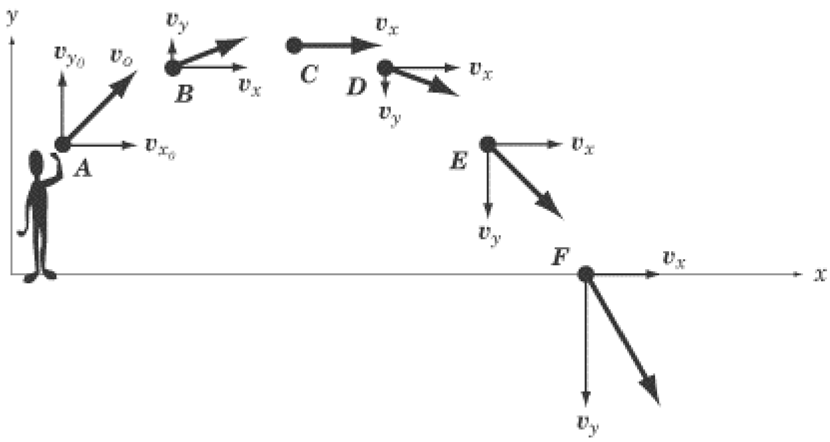
If the object is thrown/launched at an angle, you will need to use trigonometry to separate the velocity vector into its horizontal (*x*) and vertical (*y*) components:



Thus:

- horizontal velocity = $v_x = v \cos\theta$
- *initial* vertical velocity = $v_{o,y} = v \sin\theta$

Note that the vertical component of the velocity, v_y, is constantly changing because of acceleration due to gravity:



A fact worth remembering is that an angle of 45° gives the greatest horizontal displacement.

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Sample Problem:

Q: An Angry Bird* is launched upward from a slingshot at an angle of 40° with a velocity of $20\,\frac{m}{s}$. The bird strikes the pigs' fortress at the same height that it was launched from. How far away is the fortress?

A: We are looking for the horizontal distance, d_x.

We start with the equation:

$$d_x = v_x t$$

We need v_h and t.

We can substitute for v_x using $v_x = v\cos\theta$ to get:

$$d_x = (v\cos\theta)\,t = 20\cos(40°)\,t = 15.3\,t$$

We can get t from:

$$d_y = v_{o,y}t + \tfrac{1}{2}gt^2 = v(\sin\theta)\,t + \tfrac{1}{2}gt^2 = 20(\sin 40°)t + \tfrac{1}{2}(-10)t^2 = 12.9t - 5t^2$$

Because the vertical displacement is zero (the angry bird ends at the same height as it started), $d_v = 0$:

$$0 = 12.9t - 5t^2$$
$$0 = t(12.9 - 5t)$$

which has the solutions:

$$t = 0, \quad 12.9 - 5t = 0$$

The first solution ($t = 0$) is when the angry bird is launched. The second solution is the one of interest—when the angry bird lands. Solving for t gives:

$$12.9 = 5t$$
$$\frac{12.9}{5} = 2.57\,\text{s} = t$$

We can now substitute this expression into the first equation to get:

$$d_x = 15.3\,t = (15.3)(2.57) = 39.4\,\text{m}$$

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* *Angry Birds* was a video game from 2010 in which players used slingshots to shoot birds with the necessary velocity and angle to destroy a fortress and kill the bad guys, who were green pigs.

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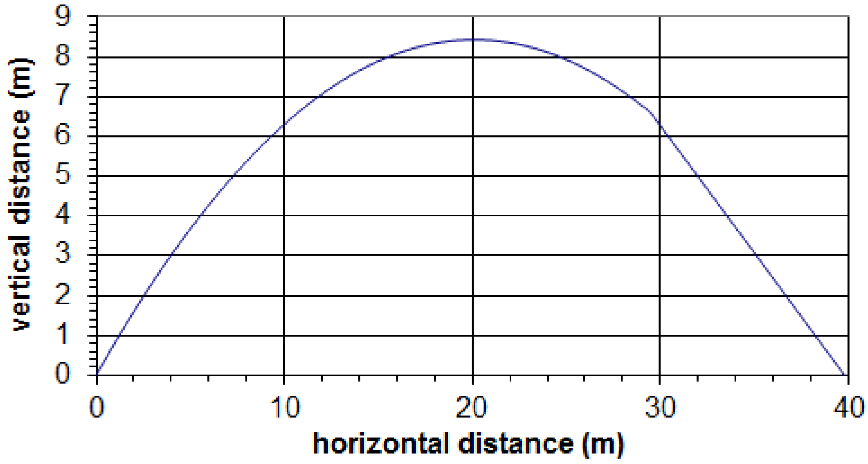
A graph of the angry bird's motion would look like the following:



If you wanted to solve this problem symbolically, you would do the following:

$$d_x = v_x t = v(\cos\theta)t$$

$$d_y = 0 = v_{o,y}t + \frac{1}{2}gt^2 = v(\sin\theta)t + \frac{1}{2}gt^2$$

$$0 = t(v\sin\theta + \frac{1}{2}gt)$$

$$v\sin\theta = -\frac{1}{2}gt$$

$$t = \frac{-2v\sin\theta}{g}$$

$$d_x = v(\cos\theta)\left(\frac{-2v\sin\theta}{g}\right) = \frac{-2v^2\sin\theta\cos\theta}{g}$$

If you have taken precalculus and you know the double angle formula, you can simplify the above expression, using $\sin 2\theta = 2\sin\theta\cos\theta$, which gives:

$$d_x = \frac{-2v^2\sin\theta\cos\theta}{g} = \frac{-v^2(2\sin\theta\cos\theta)}{g} = \frac{-v^2\sin 2\theta}{g}$$

(Of course, if you don't know the double angle formula, you can plug in the values anyway.)

Plugging in the values gives:

$$d_x = \frac{-2v\sin\theta\cos\theta}{g} = \frac{-2(20)^2(\sin 40°)(\cos 40°)}{-10} = 39.4\text{ m}$$

as before.

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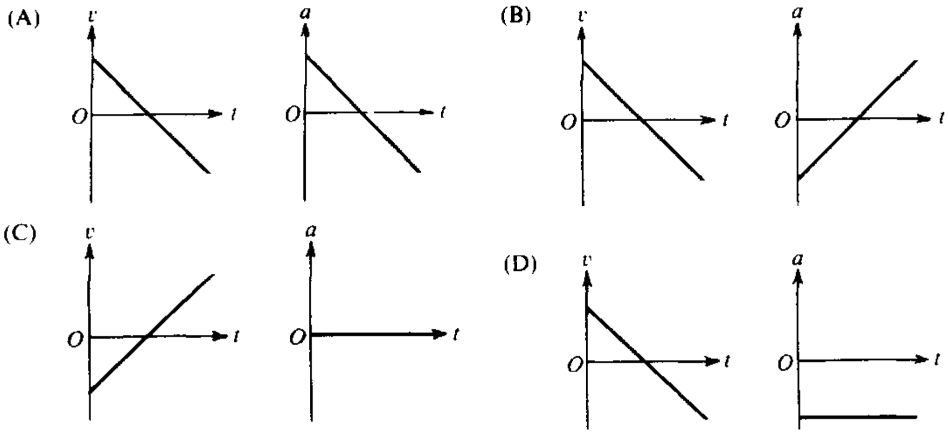
What AP® Projectile Problems Look Like

AP® motion and acceleration problems almost always involve graphs or projectiles. Here is an example that involves both:

Q:

A projectile is fired with initial velocity v_0 at an angle θ_0 with the horizontal and follows the trajectory shown above. Which of the following pairs of graphs best represents the vertical components of the velocity and acceleration, *v* and *a*, respectively, of the projectile as functions of time t?



A: *Because the object is a projectile:*

- *It can move both vertically and horizontally.*
- *It has a nonzero initial horizontal velocity. However, because the problem is asking about the vertical components, we can ignore the horizontal velocity.*
- *It has a constant acceleration of* $-g$ *(i.e., g in the downward direction) due to gravity.*

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For each pair of graphs, the first graph is velocity vs. time. The slope, $\frac{\Delta v}{\Delta t}$*, is acceleration. Because acceleration is constant, the graph has to have a constant. if we choose up to be the positive direction (which is the most common convention), correct answers would be (A), (B), and (D). If we choose down to be positive, only (C) would be correct.*

The second graph is acceleration vs. time. We know that acceleration is constant, which eliminates choices (A) and (B). We also know that acceleration is not zero, which eliminates choice (C). This leaves choice (D) as the only possible remaining answer. Choice (D) correctly shows a constant negative acceleration, because the slope of the first graph is negative, and the y-value of the second graph is also negative.

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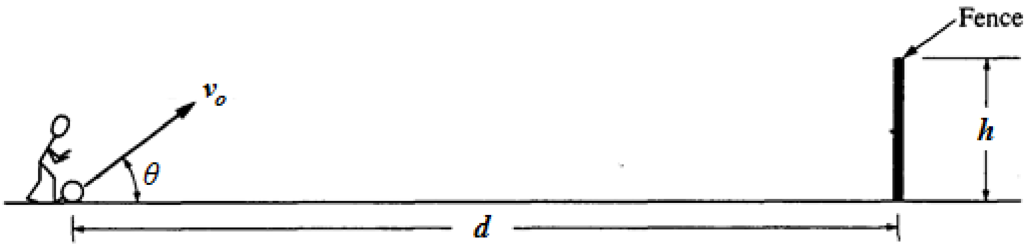
Q: A ball of mass *m*, initially at rest, is kicked directly toward a fence from a point that is a distance *d* away, as shown above. The velocity of the ball as it leaves the kicker's foot is v_o at an angle of θ above the horizontal. The ball just clears the top of the fence, which has a height of *h*. The ball hits nothing while in flight and air resistance is negligible.



a. Determine the time, *t*, that it takes for the ball to reach the plane of the fence, in terms of v_o, θ, *d*, and appropriate physical constants.

The horizontal component of the velocity is $v_h = v_o \cos\theta = \frac{d}{t}$.

Solving this expression for t gives $t = \frac{d}{v_o \cos\theta}$.

b. What is the vertical velocity of the ball when it passes over the top of the fence?

The initial vertical component of the velocity is $v_{v,o} = v_o \sin\theta$. *The equation for velocity is* $v_v = v_{o,v} + at$. *Substituting* $a = -g$, *and* $t = \frac{d}{v_o \cos\theta}$ *into this expression gives the answer of:*

$$v_v = v_o \sin\theta - \frac{gd}{v_o \cos\theta}$$

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Homework Problems

Horizontal (level) projectile problems:

1. **(M)** A diver running $1.6\,\frac{m}{s}$ dives out horizontally from the edge of a vertical cliff and reaches the water below 3.0 s later.
   a. **(M)** How high was the cliff?

      Answer: 45 m

   b. **(M)** How far from the base did the diver hit the water?

      Answer: 4.8 m

2. **(S)** A ball is thrown horizontally from the roof of a building 56 m tall and lands 45 m from the base. What was the ball's initial speed?

   Answer: $13.4\,\frac{m}{s}$

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3. **(M – honors & AP®; A – CP1)** A tiger leaps horizontally from a rock with height *h* at a speed of v_o. What is the distance, *d*, from the base of the rock where the tiger lands?

   (*If you are not sure how to do this problem, do #4 below and use the steps to guide your algebra.*)

   Answer: $d = v_o\sqrt{\frac{2h}{g}}$

4. **(S – honors & AP®; M – CP1)** A tiger leaps horizontally from a 7.5 m high rock with a speed of $4.5\,\frac{\text{m}}{\text{s}}$. How far from the base of the rock will he land?

   (<u>You must start with the equations in your Physics Reference Tables *and show all of the steps of GUESS.*</u> *You may only use the answer to question #3 above as a starting point if you have already solved that problem.*)

   Answer: 5.5 m

5. **(M)** The pilot of an airplane traveling $45\,\frac{\text{m}}{\text{s}}$ wants to drop supplies to flood victims isolated on a patch of land 160 m below. The supplies should be dropped when the plane is how far from the island?

   Answer: 255 m

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Problems involving projectiles launched at an angle:

6. **(M – honors & AP®; A – CP1)** A ball is shot out of a slingshot with a velocity of $10.0\,\frac{m}{s}$ at an angle of 40.0° above the horizontal. How far away does it land?

   Answer: 9.85 m

7. **(S – honors & AP®; A – CP1)** The 12 Pounder Napoleon Model 1857 was the primary cannon used during the American Civil War. If the cannon had a muzzle velocity of $439\,\frac{m}{s}$ and was fired at a 5.00° angle, what was the effective range of the cannon (the distance it could fire)? (Neglect air resistance.)

   Answer: 3 347 m (Note that this is more than 2 miles!)

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8. **(M – AP®; S – honors; A – CP1)** A physics teacher is designing a ballistics event for a science competition. The ceiling is 3.00 m high, and the maximum velocity of the projectile will be $20.0\,\frac{m}{s}$.

   a. What is the maximum that the vertical component of the projectile's initial velocity could have?

   Answer: $7.75\,\frac{m}{s}$

   b. At what angle should the projectile be launched in order to achieve this maximum height?

   Answer: 22.8°

   c. What is the maximum horizontal distance that the projectile could travel?

   Answer: 28.6 m

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