CP1 & honors (not AP®)

Length Contraction & Time Dilation

Unit: Special Relativity

NGSS Standards/MA Curriculum Frameworks (2016): N/A

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.D.3.1

Mastery Objective(s): (Students will be able to…)

- Explain how and why distance and time change at relativistic speeds.

Success Criteria:

- Explanations account for observed behavior.

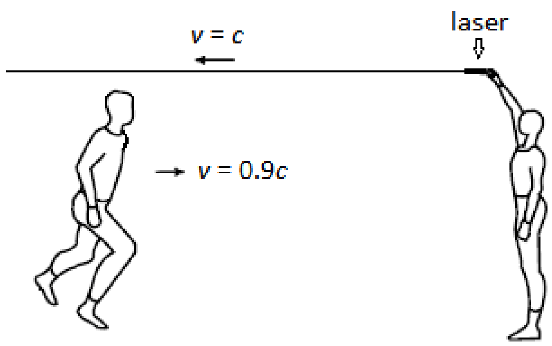
Language Objectives:

- Discuss how length contraction and/or time dilation can lead to a paradox.

Tier 2 Vocabulary: reference frame, contraction, dilation

Notes:

Based on Maxwell's conclusions, if an observer were somehow running with a relativistic speed of light toward an oncoming beam of light, the student should measure the same velocity of light as a stationary observer:



However, the amount of time it takes for a photon of light to pass the moving observer must be significantly less than the amount of time it would take a photon to pass a stationary observer.

Because velocity depends on distance and time, if the velocity of light cannot change, then as the observer approaches the speed of light, this means the distance and/or time must change!

For most people, the idea that distance and time depend on the reference frame is just as strange and uncomfortable as the idea that the speed of light cannot depend on the reference frame.

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Length Contraction

If an object is moving at relativistic speeds and the velocity of light must be constant, then distances must become shorter as velocity increases. This means that as the velocity of an object approaches the speed of light, distances in its reference frame approach zero.

The Dutch physicist Hendrick Lorentz determined that the apparent change in length should vary according to the formula:

$$L = L_o\sqrt{1 - \frac{v^2}{c^2}}$$

where:

- L = length of moving object
- L_o = "proper length" of object (length of object at rest)
- v = velocity of object
- c = velocity of light

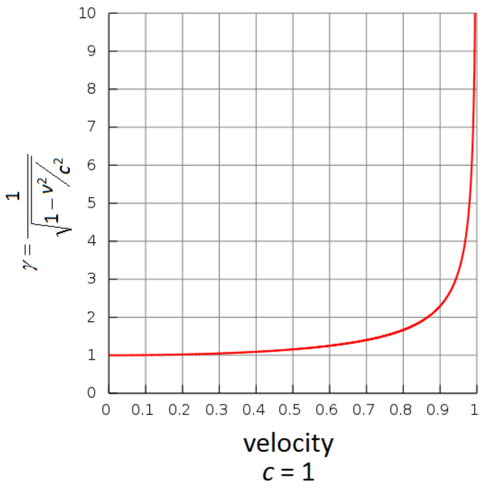
The ratio of L_o to L is named after Lorentz and is called the Lorentz factor (γ):

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The contracted length is therefore given by the equations:

$$L = \frac{L_o}{\gamma} \qquad \text{or} \qquad \frac{L_o}{L} = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The Lorentz factor, γ, is 1 at rest and approaches infinity as the velocity approaches the speed of light:



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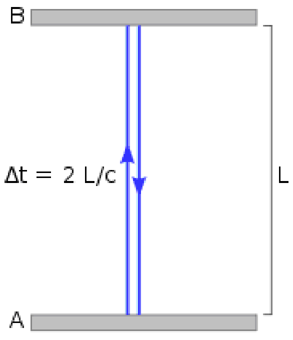
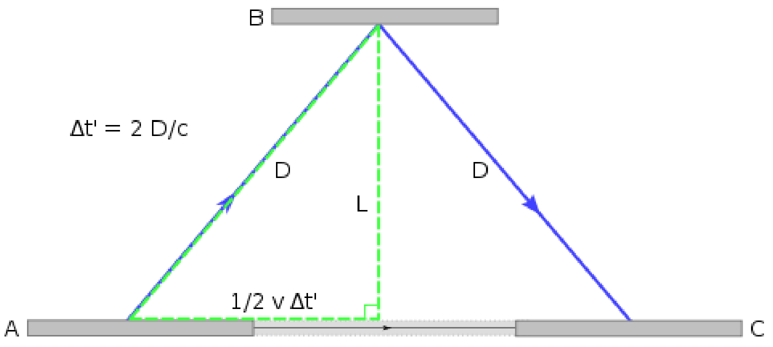
Time Dilation

In order to imagine how time is affected at relativistic speeds, imagine a clock that keeps time by sending a pulse of light from point (A), bouncing it off a mirror at point (B) and then measuring the time it takes to reach a detector back at point (A). The distance between the two surfaces is L, and the time for a pulse of light to travel to the mirror and back is therefore $\Delta t = \frac{2L}{c}$.



However, suppose the clock is moving at a relativistic speed. In the moving reference frame the situation looks exactly like the situation above. However, from an inertial (stationary) reference frame, the situation would look like the following:



Notice that in the stationary reference frame, the pulses of light must travel farther because of the motion of the mirror and detector. Because the speed of light is constant, the longer distance takes a longer time.

In other words, time is longer in the inertial (stationary) reference frame than it is in the moving reference frame!

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This conclusion has significant consequences. For example, events that happen in two different locations could be simultaneous in one reference frame, but occur at different times in another reference frame!

Using arguments similar to those for length contraction, the equation for time dilation turns out to be:

$$\Delta t' = \gamma\, \Delta t \qquad \text{or} \qquad \frac{\Delta t'}{\Delta t} = \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

where:

$\Delta t'$ = time difference between two events in stationary reference frame
Δt = time difference between two events in moving reference frame
v = velocity of moving reference frame
c = velocity of light

Effect of Gravity on Time

Albert Einstein first postulated the idea that gravity slows down time in his paper on special relativity. This was confirmed experimentally in 1959.

As with relativistic time dilation, gravitational time dilation relates a duration of time in the absence of gravity ("proper time") to a duration in a gravitational field. The equation for gravitational time dilation is:

$$\Delta t' = \Delta t \sqrt{1-\frac{2GM}{rc^2}}$$

where:

$\Delta t'$ = time difference between two events in stationary reference frame
Δt = time difference between two events in moving reference frame
G = universal gravitational constant $(6.67\times 10^{-11}\ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2})$
M = mass of the object creating the gravitational field
r = observer's distance (radius) from the center of the massive object
c = velocity of light

In 2014, a new atomic clock was built at the University of Colorado at Boulder, based on the vibration of a lattice of strontium atoms in a network of crisscrossing laser beams. The clock has been improved even since its invention, and is now accurate to better than one second per fifteen billion years (the approximate age of the universe). This clock is precise enough to measure differences in time caused by differences in the gravitational pull of the Earth near Earth surface. This clock would run measurably faster on a shelf than on the floor, because of the differences in time itself due to the Earth's gravitational field.

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Black Holes

A black hole is an object that is so dense that its escape velocity (See *Escape Velocity* starting on page 481) is faster than the speed of light, which means even light cannot escape.

For this to happen, the radius of the black hole needs to be smaller than $\frac{2GM}{c^2}$.

This results in a negative value for $\sqrt{1-\frac{2GM}{rc^2}}$, which makes $\Delta t'$ imaginary.

The consequence of this is that time is imaginary (does not pass) on a black hole, and therefore light cannot escape. This critical value for the radius is called the Schwarzschild radius, named for the German astronomer Karl Schwarzschild who first solved Einstein's field equations exactly in 1916 and postulated the existence of black holes.

The Sun is too small to be able to form a black hole, but if it were collapsed to the density of a black hole, its Schwarzschild radius would be approximately 3.0 km. If the Earth were collapsed to the density of a black hole, its radius would be approximately 9.0 mm.

Sample problem:

Q: In the science fiction TV show *Star Trek*, in order to avoid detection by the Borg, the starship Enterprise must make itself appear to be less than 25 m long. If the rest length of the Enterprise is 420 m, how fast must it be traveling? What fraction of the speed of light is this?

A: $L = 25\text{ m}$

$L_o = 420\text{ m}$

$c = 2.998\times10^8\ \frac{\text{m}}{\text{s}}$

$$\gamma=\frac{L_o}{L}=\frac{1}{\sqrt{1-v^2/c^2}}$$

$$\frac{420}{25}=\frac{1}{\sqrt{1-v^2/(2.998\times10^8)^2}}$$

$$\frac{25}{420}=0.0595=\sqrt{1-v^2/(2.998\times10^8)^2}$$

$$(0.0595)^2=0.00354=1-\frac{v^2}{8.988\times10^{16}}$$

$$\frac{v^2}{8.988\times10^{16}}=1-0.00354=0.99646$$

$$v^2=(0.99646)(8.988\times10^{16})$$

$$v^2=8.956\times10^{16}$$

$$v=\sqrt{8.956\times10^{16}}=2.993\times10^8\ \tfrac{\text{m}}{\text{s}}$$

$$\frac{2.993\times10^8}{2.998\times10^8}=0.998\,c$$

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Homework Problems

1. **(M)** A spaceship is travelling at 0.7*c* on a trip to the Andromeda galaxy and returns to Earth 25 years later (from the reference frame of the people who remain on Earth). How many years have passed for the people on the ship?

   Answer: 17.85 years

2. **(S)** A 16-year-old girl sends her 48-year-old parents on a vacation trip to the center of the universe. When they return, the parents have aged 10 years, and the girl is the same age as her parents. How fast was the ship going? (*Give your answer in terms of a fraction of the speed of light.*)

   Answer: 0.971*c*

3. **(M)** The starship Voyager has a length of 120 m and a mass of 1.30×10^6 kg at rest. When it is travelling at $2.88 \times 10^8 \frac{m}{s}$, what is its apparent length according to a stationary observer?

   Answer: 33.6 m