CP1 & honors (not AP®)

Energy-Momentum Relation

Unit: Special Relativity

NGSS Standards/MA Curriculum Frameworks (2016): N/A

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): N/A

Mastery Objective(s): (Students will be able to…)

- Explain how and why mass and momentum change at relativistic speeds.

Success Criteria:

- Explanations account for observed behavior.

Language Objectives:

- Discuss how length contraction and/or time dilation can lead to a paradox.

Tier 2 Vocabulary: reference frame, contraction, dilation

Notes:

The momentum of an object also changes according to the Lorentz factor as it approaches the speed of light:

$$p = \gamma\, p_o \qquad \text{or} \qquad \frac{p}{p_o} = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where:

p = momentum of object in moving reference frame

p_o = momentum of object in stationary reference frame

v = velocity of moving reference frame

c = velocity of light

Because momentum is conserved, an object's momentum in its own reference frame must remain constant. Therefore, at relativistic speeds the object's mass must change!

The equation for relativistic mass is:

$$m = \gamma\, m_o \qquad \text{or} \qquad \frac{m}{m_o} = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where:

m = mass of object in moving reference frame

m_o = mass of object at rest

Therefore we can write the momentum equation as:

$$p = \gamma\, m_o v$$

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Note that as the velocity of the object approaches the speed of light, the denominator of the Lorentz factor, $\sqrt{1-\frac{v^2}{c^2}}$ approaches zero, which means that the Lorentz factor approaches infinity.

Therefore the momentum of an object must also approach infinity as the velocity of the object approaches the speed of light.

This relationship creates a potential problem. An object with infinite momentum must have infinite kinetic energy, but Einstein's equation $E = mc^2$ is finite. While it is true that the relativistic mass becomes infinite as velocity approaches the speed of light, there is still a discrepancy. Recall from mechanics that:

$$E_k = \frac{p^2}{2m}$$

According to this formula, the energy predicted using relativistic momentum should increase faster than the energy predicted by using $E = mc^2$ with relativistic mass. Obviously the amount of energy cannot depend on how the calculation is performed; the problem must therefore be that Einstein's equation needs a correction for relativistic speeds.

The solution is to modify Einstein's equation by adding a momentum term. The resulting energy-momentum relation is:

$$E^2 = (pc)^2 + (mc^2)^2$$

This equation gives results that are consistent with length contraction, time dilation and relativistic mass.

For an object at rest, its momentum is zero, and the equation reduces to the familiar form:

$$E^2 = 0 + (mc^2)^2$$
$$E = mc^2$$