Vectors

Unit: Mathematics

NGSS Standards/MA Curriculum Frameworks (2016): SP5

AP® Physics 1 Learning Objectives/Essential Knowledge (2024): 1.1.A.1, 1.1.A.2, 1.1.A.3, 1.1.A.3.i, 1.1.A.3.ii, 1.1.B.1, 1.5.A, 1.5.A.1, 1.5.A.2, 1.5.A.3

Mastery Objective(s): (Students will be able to…)

- Identify the magnitude and direction of a vector.
- Combine vectors graphically and calculate the magnitude and direction.

Success Criteria:

- Magnitude is calculated correctly (Pythagorean theorem).
- Direction is correct: angle (using trigonometry) or direction (*e.g.,* "south", "to the right", "in the negative direction", *etc.*)

Language Objectives:

- Explain what a vector is and what its parts are.

Tier 2 Vocabulary: magnitude, direction

Notes:

<u>vector</u>: a quantity that has a direction as well as a magnitude (value/quantity).

E.g., if you are walking $1\frac{\text{m}}{\text{s}}$ to the north, the magnitude is $1\frac{\text{m}}{\text{s}}$ and the direction is north.

<u>scalar</u>: a quantity that has a value/quantity but does not have a direction. (A scalar is what you think of as a "regular" number, including its unit.)

<u>magnitude</u>: the part of a vector that is not the direction (*i.e.,* the value including its units). If you have a force of 25 N to the east, the magnitude of the force is 25 N.

The mathematical operation of taking the magnitude of a vector is represented by two double vertical bars (like double absolute value bars) around the vector. For example, if $\vec{\boldsymbol{F}}$ is 25 N to the east, then $\|\vec{\boldsymbol{F}}\| = 25\,\text{N}$. However, to make typesetting easier, it is common to use regular absolute value bars instead, *e.g.,* $|\vec{F}| = 25\,\text{N}$.

<u>resultant</u>: a vector that is the result of a mathematical operation (such as the addition of two vectors).

Variables that represent vectors are traditionally typeset in ***bold Italics***. Vector variables may also optionally have an arrow above the letter:

$$\boldsymbol{J}, \vec{\boldsymbol{F}}, \boldsymbol{v}$$

Variables that represent scalars are traditionally typeset in *plain Italics:*

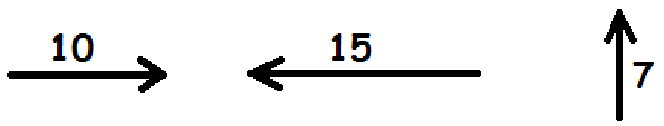
$$V,\ t,\ \lambda$$

Variable that represent only the magnitude of a vector (*e.g.,* in equations where the direction is not relevant) are typeset as if they were scalars:

For example, suppose $\vec{\boldsymbol{F}}$ is a vector representing a force of 25 N to the east. (Notice that the vector includes the magnitude or amount ***and*** the direction.)

If we needed a variable to represent only the magnitude of 25 N, we would use the variable *F*.

Vectors are represented graphically using arrows. The length of the arrow represents the magnitude of the vector, and the direction of the arrow represents the direction of the vector:



magnitude 10	magnitude 15	magnitude 7
direction: "to the right", 0°	direction: "to the left", +180°	direction: "up", +90°

The negative of a vector is a vector with the same magnitude in the opposite direction:

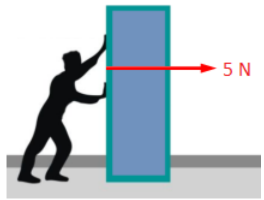
Note, however, that we use positive and negative numbers to represent the direction of a vector, but a negative value for a vector does not mean the same thing as a negative number in mathematics. In math, $-10 < 0 < +10$, because positive and negative numbers represent locations on a continuous number line.

However, a *velocity* of $-10\,\frac{\text{m}}{\text{s}}$ means "$10\,\frac{\text{m}}{\text{s}}$ in the negative direction". This means that $-10\,\frac{\text{m}}{\text{s}} > +5\,\frac{\text{m}}{\text{s}}$, because the first object is moving faster than the second ($10\,\frac{\text{m}}{\text{s}}$ *vs.* $5\,\frac{\text{m}}{\text{s}}$), even though the objects are moving in opposite directions.

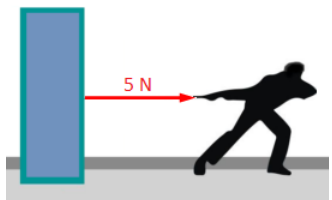
Translating Vectors

Vectors have a magnitude and direction but not a location. This means we can translate a vector (in the geometry sense, which means to move it without changing its size or orientation), and it's still the same vector quantity.

For example, consider a person pushing against a box with a force of 5 N to the right. We will define the positive direction to be to the right, which means we can call the force +5 N:

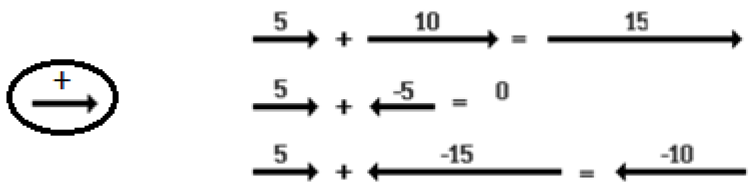


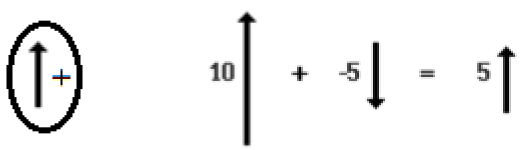
If the force is moved to the other side of the box, it's still 5 N to the right (+5 N), which means it's still the same vector:



Adding Vectors in One Dimension

If you are combining vectors in one dimension (*e.g,* horizontal), adding vectors is just adding positive and/or negative numbers:

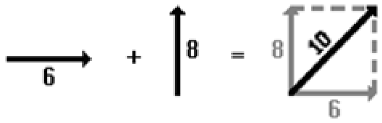




Adding vectors in Two Dimensions

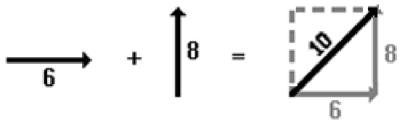
If the vectors are not in the same direction, we translate (slide) them until they meet, either tail-to-tail or tip-to-tail, and complete the parallelogram.

If the vectors are at right angles to one another, the parallelogram is a rectangle and we can use the Pythagorean theorem to find the magnitude of the resultant:



$(6^2 + 8^2 = 10^2)$

tail-to-tail



$(6^2 + 8^2 = 10^2)$

tip-to-tail

Note that the sum of these two vectors has a magnitude (length) of 10, not 14; "Adding" vectors means combining them using geometry.

The vector sum comes out the same whether you combine the vectors tail-to-tail or tip-to-tail. The decision of how to represent the vectors depends on the situation that you are modeling with them:

- Two forces pulling on the same object (think of two ropes connected to the same point) is best represented by drawing the vectors tail-to-tail.
- The displacement* of a walking path that starts in one direction and then turns is best represented by drawing the vectors tip-to-tail.

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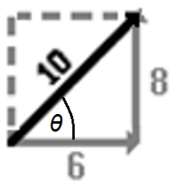
When working with vectors in multiple dimensions, we can use positive and negative numbers if the vectors are along the *x*-axis or the *y*-axis, but we need to use an angle to specify any other direction. To calculate the direction of the resultant of the vector operation above, we need to use trigonometry:



$$\tan\theta = \frac{8}{6} \quad \rightarrow \quad \theta = \tan^{-1}\left(\frac{8}{6}\right) = 53.1°$$

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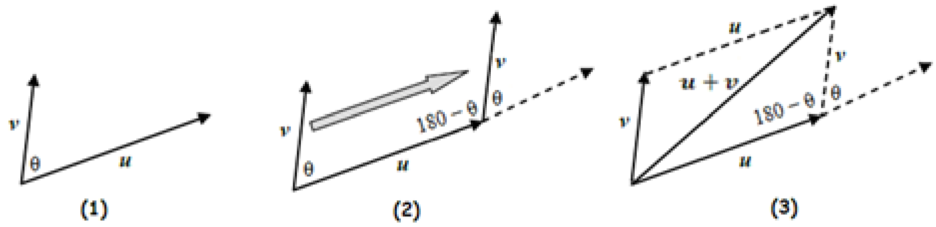
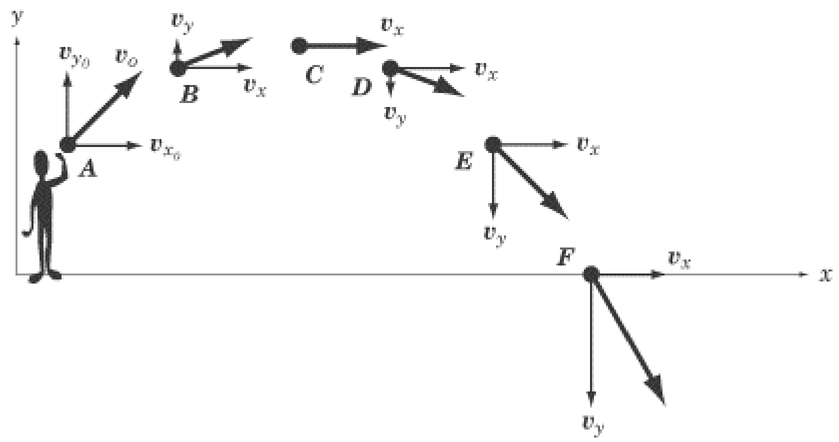
* Displacement is a vector quantity that represents the straight-line distance from one point to another. Displacement is covered in the next unit, Kinematics in One Dimension.

We would use exactly the same process to add vectors that are not perpendicular:



The trigonometry needed for these calculations requires the laws of sines and cosines. The calculations are not difficult, but this use of trigonometry is beyond the scope of this course.

One type of physics problem that commonly uses vectors is two-dimensional projectile motion. If the motion of the projectile is represented by a vector, $\vec{\boldsymbol{v}}$, at angle θ, we can split the vector into a horizontal component, which we call $\vec{\boldsymbol{v}}_x$ and a vertical component, which we call $\vec{\boldsymbol{v}}_y$.



Notice that, in the case of projectile motion (such as throwing a ball), $\vec{\boldsymbol{v}}_x$ remains constant, but $\vec{\boldsymbol{v}}_y$ changes (because of the effects of gravity).

Because perpendicular vectors do not affect each other, we can apply equations to the two directions separately.

For example, in projectile motion (which you will learn about in detail in the **Error! Reference source not found.** topic starting **Error! Bookmark not defined.**), we usually use the equation $\vec{\boldsymbol{d}} = \vec{\boldsymbol{v}}_o t + \frac{1}{2}\vec{\boldsymbol{a}}t^2$, applying it separately in the *x*- and *y*-directions. This gives us two equations.

In the horizontal (*x*)-direction:

$$\vec{\boldsymbol{d}}_x = \vec{\boldsymbol{v}}_{o,x} t + \tfrac{1}{2}\cancelto{0}{\vec{\boldsymbol{a}}_x} t^2$$
$$\vec{\boldsymbol{d}}_x = \vec{\boldsymbol{v}}_x t$$

In the vertical (*y*)-direction:

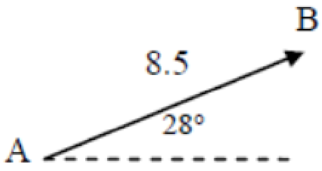
$$\vec{\boldsymbol{d}}_y = \vec{\boldsymbol{v}}_{o,y} t + \frac{1}{2}\vec{\boldsymbol{a}}_y t^2$$
$$\vec{\boldsymbol{d}}_y = \vec{\boldsymbol{v}}_{o,y} t + \frac{1}{2}\vec{\boldsymbol{g}} t^2$$

Note that each of the vector quantities ($\vec{\boldsymbol{d}}, \vec{\boldsymbol{v}}_o$ and $\vec{\boldsymbol{a}}$) has independent *x*- and *y*-components. For example, $\vec{\boldsymbol{v}}_{o,x}$ (the component of the initial velocity in the *x*-direction) is independent of $\vec{\boldsymbol{v}}_{o,y}$ (the component of the initial velocity in the *x*-direction). This means *we treat them as completely separate variables*, and we can solve for one without affecting the other.

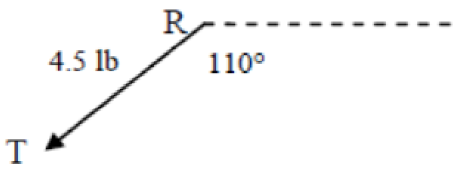
Homework Problems

Label the magnitude and direction (relative to horizontal) of each of the following:

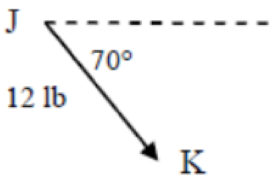
1. **(M)**



2. **(M)**

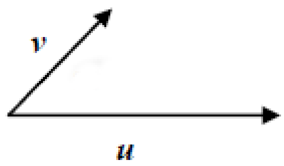


3. **(S)**

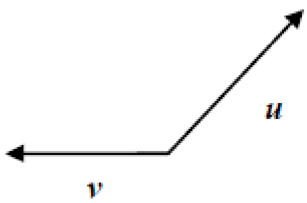


Sketch the resultant of each of the following.

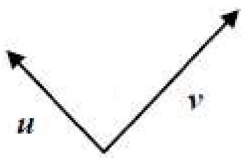
4. **(M)**



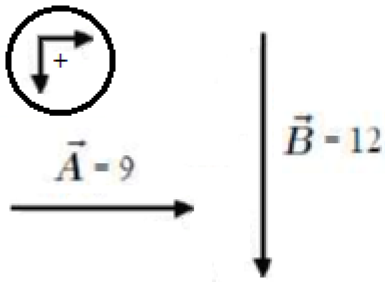
5. **(M)**



6. **(S)**



Consider the following vectors $\vec{A} \& \vec{B}$.



Vector $\vec{A}$ has a magnitude of 9 and its direction is the positive horizontal direction (to the right).

Vector $\vec{B}$ has a magnitude of 12 and its direction is the positive vertical direction (down).

7. **(M)** Sketch the resultant of $\vec{A}+\vec{B}$, and determine its magnitude and direction*.

8. **(S)** Sketch the resultant of $\vec{A}-\vec{B}$ (which is the same as $\vec{A}+(-\vec{B})$, and determine its magnitude and direction*.

* Finding the direction requires trigonometry. If your teacher skipped the right-angle trigonometry section, you only need to find the magnitude.