honors (not AP®)

Uncertainty & Error Analysis

Unit: Laboratory & Measurement

MA Curriculum Frameworks (2016): SP4

AP® Physics 2 Learning Objectives: SP5

Mastery Objective(s): (Students will be able to…)

- Determine the uncertainty of a measured or calculated value.

Success Criteria:

- Take analog measurements to one extra digit of precision.
- Correctly estimate measurement uncertainty.
- Correctly read and interpret stated uncertainty values.
- Correctly propagate uncertainty through calculations involving addition/subtraction and multiplication/division.

Tier 2 Vocabulary: uncertainty, error

Language Objectives:

- Understand and correctly use the terms "uncertainty" and "relative error."
- Correctly explain the process of estimating and propagating uncertainty.

Tier 2 Vocabulary: uncertainty, error

Notes:

In science, unlike mathematics, there is no such thing as an exact answer. Ultimately, every quantity is limited by the precision and accuracy of the measurements that it came from. If you can only measure a quantity to within 10 %, that means any calculation that is derived from that measurement can't be any better than ±10 %.

Error analysis is the practice of determining and communicating the causes and extents of uncertainty in your results. Error analysis involves understanding and following the uncertainty in your data, from the initial measurements to the final calculated and reported results.

Note that the word "error" in science has a different meaning from the word "error" in everyday language. In science, **"error" means "uncertainty."** If you report that you drive (2.4 ± 0.1) miles to school every day, you would say that this distance has an error of ±0.1 mile. This does not mean your car's odometer is wrong; it means that the actual distance *could be* 0.1 mile more or 0.1 mile less—*i.e.,* somewhere between 2.3 and 2.5 miles. ***When you are analyzing your results, <u>never</u> use the word "error" to mean mistakes that you might have made!***

Use this space for summary and/or additional notes:

honors (not AP®)

Uncertainty

The uncertainty or error of a measurement describes how close the actual value is likely to be to the measured value. For example, if a length was measured to be 22.3 cm, and the uncertainty was 0.3 cm (meaning that the measurement is only known to within ± 0.3 cm), we could represent this measurement in either of two ways:

22.3 ± 0.3 cm[2]   22.3(3) cm

The first of these states the variation (±) explicitly in cm (the actual unit). The second is shows the variation in the last digits shown.

What it means is that the true length is approximately 22.3 cm, and is statistically likely[3] to be somewhere between 22.0 cm and 22.6 cm.

Absolute Error

Absolute error (or absolute uncertainty) refers to the uncertainty in the actual measurement. For the measurement 22.3 ± 0.3 cm, the absolute error is ± 0.3 cm.

Relative Error

Relative error shows the error or uncertainty as a fraction of the total.

The formula for relative error is $\text{R.E.} = \dfrac{\text{uncertainty}}{\text{measured value}}$

For the measurement 22.3 ± 0.3 cm, the relative error would be 0.3 cm out of 22.3 cm. Mathematically, we express this as:

$$\text{R.E.} = \frac{0.3\,\cancel{\text{cm}}}{22.3\,\cancel{\text{cm}}} = 0.013$$

Note that relative error is dimensionless (does not have any units), because the numerator and denominator have the same units, which means the units cancel.

Percent Error

Percent error is relative error expressed as a percentage. You can turn relative error into percent error by multiplying by 100.

In the example above, the relative error of 0.013 would be 1.3 % error.

---

[2] The unit is assumed to apply to both the value and the uncertainty. The unit for the value and uncertainty should be the same. A value of 10.63 m ± 2 cm should be rewritten as 10.63 ± 0.02 m

[3] Statistically, the standard uncertainty is one standard deviation, which is discussed on the following page.

Use this space for summary and/or additional notes:

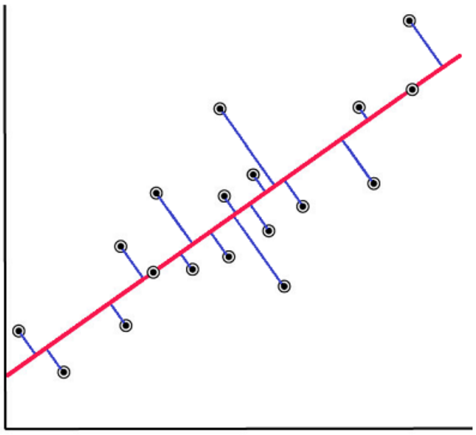
honors (not AP®)

Best-Fit Lines & Standard Deviation

best-fit line: a line that represents the expected value of your dependent variable for values of your dependent variable. The best-fit line minimizes the total accumulated error (difference between each actual data point and the line).

standard deviation (σ): the average of how far each data point is from its expected value.

The standard deviation is calculated mathematically as the average difference between each data point and the value predicted by the best-fit line.

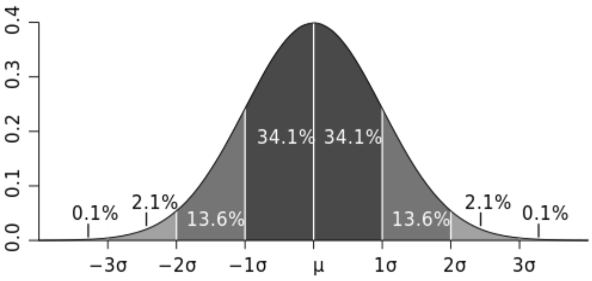


A small standard deviation means that most or all of the data points lie close to the best-fit line. A larger standard deviation means that on average, the data points lie farther from the line.

Unless otherwise stated, ***the standard deviation is the uncertainty (the "plus or minus") of a calculated quantity***. *E.g.,* a measurement of 25.0 cm with a standard deviation of 0.5 cm would be expressed as (25.0 ± 0.5) cm.

The expected distribution of values relative to the mean is called the Gaussian distribution (named after the German mathematician Carl Friedrich Gauss.)

It looks like a bell, and is often called a "bell curve".



Statistically, approximately 68 % of the measurements are expected to fall within one standard deviation of the mean, *i.e.,* within the standard uncertainty.

correlation coëfficient (*R* or R^2 value): a measure of how linear the data are—how well they approximate a straight line. In general, an R^2 value of less than 0.9 means that the data are not linear, there was a problem with one or more data points, or there was a problem with the entire experiment.

Use this space for summary and/or additional notes:

honors (not AP®)

Uncertainty of Measurements

If you have the ability to measure a quantity that is not changing (such as the mass or length of an object), you will get the same value every time you measure it. This means you have only one data point.

When you have only one data point, the uncertainty is the limit of how well you can measure it. This will be your best educated guess, based on how closely you think you actually measured the quantity. This means you need to take measurements as carefully and precisely as possible, because *every careless measurement needlessly increases the uncertainty of the result*.

Digital Measurements

For digital equipment, if the reading is *stable* (not changing), look up the published precision of the instrument in its user's manual. (For example, many balances used in high schools have a readability of 0.01 g but are only precise to within ± 0.02 g.) If there is no published value (or the manual is not available), assume the uncertainty is ± 1 in the last digit.

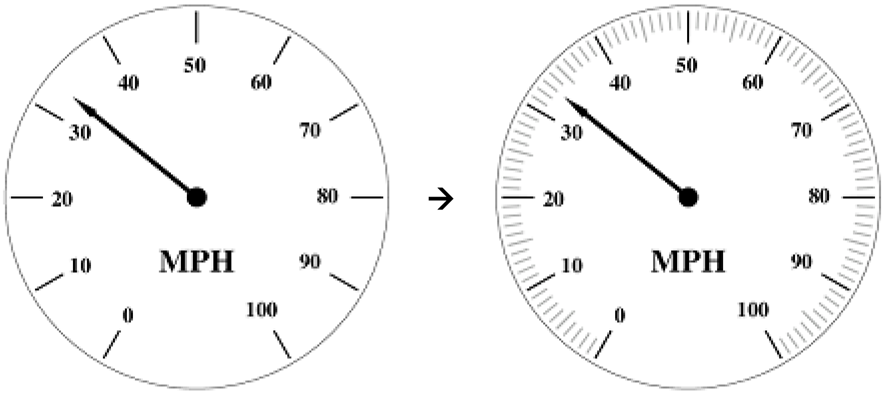
If the reading is *unstable* (changing), state the reading as the average of the highest and lowest values, and the uncertainty as the amount that you would need to add to or subtract from the average to obtain either of the extremes (but never less than the published uncertainty of the equipment).

Use this space for summary and/or additional notes:

honors (not AP®)

Analog Measurements

When making analog measurements, *always* estimate one extra digit beyond the finest markings on the equipment. For example, if you saw the speedometer on the left, you would imagine that each tick mark was divided into ten smaller tick marks like the one on the right.



what you see: between 30 & 40 MPH

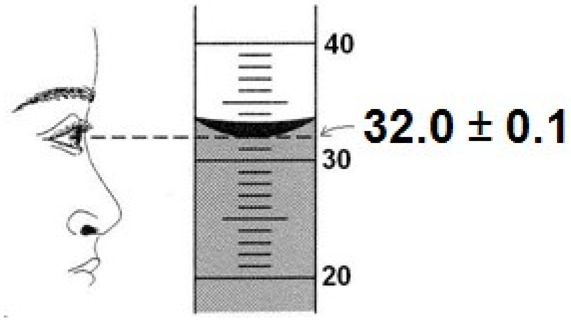
what you visualize: 33 ± 1 MPH

Note that the ***measurement and uncertainty must be expressed to the same decimal place***.

Use this space for summary and/or additional notes:

honors (not AP®)

For example, in the diagram below, the graduated cylinder is marked in 1 mL increments. When measuring volume in this graduated cylinder[4], you would estimate and write down the volume to the nearest 0.1 mL, as shown:



In the above experiment, you *must* record the volume as:

32.0 ± 0.1 mL ← correct

32 ± 0.1 mL ← wrong

32 ± 1 mL ← inadequate

In other words, the zero at the end of 32.0 mL is *required*. It is necessary to show that *you measured the volume to the nearest tenth, not to the nearest one.*

When estimating, the uncertainty depends on how well you can see the markings, but you can usually assume that the estimated digit has an uncertainty of $\pm \frac{1}{10}$ of the finest markings on the equipment. Here are some examples:

Equipment	Typical Markings	Estimate To	Assumed Uncertainty
ruler	1 mm	0.1 mm	± 0.1 mm
25 mL graduated cylinder	0.2 mL	0.02 mL	± 0.02 mL
thermometer	1 °C	0.1 °C	± 0.1 °C

[4] Remember that for most liquids, which have a downward meniscus, volume is measured at the *bottom* of the meniscus.

Use this space for summary and/or additional notes:

honors (not AP®)

Propagating Uncertainty in Calculations

When you perform calculations using numbers that have uncertainty, you need to propagate the uncertainty through the calculation.

Addition & Subtraction

When quantities with uncertainties are added or subtracted, add the quantities to get the answer, then add the uncertainties to get the total uncertainty.

Sample Problem:

Q: A substance is being heated. You record the initial temperature as (23.0 ± 0.2) °C, and the final temperature as (84.4 ± 0.2) °C. You need to calculate the temperature change (Δ*T*) with its uncertainty to use in a later calculation. What is the temperature change?

A: To calculate Δ*T*, simply subtract:

$$\Delta T = T_{final} - T_{initial} = 84.4 - 23.0 = 61.4\,°\text{C}$$

To calculate the uncertainty, add the individual uncertainties (even though the quantities were subtracted):

$$u = 0.2 + 0.2 = 0.4\,°\text{C}$$

Report the value as: $\Delta T = (61.4 \pm 0.4)\,°\text{C}$

Multiplication & Division

Because most calculations that we will perform in physics involve multiplication and/or division, you can

For calculations involving multiplication and division, estimate the uncertainty of your calculated answer by adding the relative errors and applying the total relative error to your result.

1. Perform the calculation for the desired quantity.
2. Divide the uncertainty (the ±) for each quantity by its measured value to determine its relative error.

$$\text{R.E.} = \frac{\text{uncertainty}}{\text{measured value}}$$

3. Add up all of the relative errors to get the total relative error.
4. Multiply your calculated result by the total relative error to get its uncertainty (the ± amount).

Use this space for summary and/or additional notes:

honors (not AP®)

Note: ***Most of the calculations that you will perform in physics involve multiplication and/or division, so almost all of your uncertainty calculations throughout the course will use relative error.***

Exponents

Calculations that involve ***exponents*** use the same rule as for multiplication and division. If you think of exponents as multiplying a number by itself the indicated number of times, it means you would need to add the relative error of that number that many times.

In other words, when a value is raised to an exponent, multiply its relative error by the exponent.

Sample Problem:

Q: You want to determine the amount of heat released by a process. You use the heat from the reaction to heat up some water in an insulated container called a calorimeter. You will calculate the heat using the equation: $Q = mC\Delta T$.

Suppose you recorded the following data (including uncertainties):

- The mass of the water in the calorimeter is (24.8 ± 0.1) g.
- The temperature change of the water was (12.4± 0.2) °C.
- The specific heat capacity of water is $4.18 \frac{\text{J}}{\text{g•°C}}$. (This is a published value. The uncertainty of this value is so small that we can leave it out of our calculations.)

A: The heat released by the reaction is given by the equation:

$$Q = mC\Delta T$$
$$Q = (24.8)(4.18)(12.4)$$
$$Q = 1\,285.43 \text{ J}$$

The relative errors for the two quantities that we measured are:

- mass: $\frac{0.1}{24.8} = 0.004\,03$
- temperature change: $\frac{0.2}{12.4} = 0.016\,13$

The total relative error is 0.004 03 + 0.016 13 = 0.020 16

The uncertainty is therefore (0.020 16)(1 285.43) = ±25.92 J

(Note that the absolute uncertainty has the same units as the measurement.)

We would report the measurement as (1 285.43 ± 25.92) J.

Use this space for summary and/or additional notes:

honors (not AP®)

Rounding

In the example above, the uncertainty tells us that our actual result could be different from our calculated value by as much as 25.92 J.

However, we only estimated one digit (which happened to be the tenths place) when we took our measurements. This means we have only one digit of uncertainty. Because we can't report more precision than we actually have, we need to round the calculated uncertainty off, so that we have only one unrounded digit. This means we should report our uncertainty as ± 30 J.

It wouldn't make sense to report our answer as (1 285.43 ± 30) J. Think about that—if the *tens* digit could be different from our calculated value, there is no point in reporting the ones or tenths digits. So we need to round our calculated answer to the same place value as the uncertainty—the tens place.

This means our final, rounded answer should be (1 290 ± 30) J.

Use this space for summary and/or additional notes:

Honors (not AP®)

Sample Problem #2:

Q: You need to find the density of a piece of metal. We measure its mass on a balance to be (24.75 ± 0.02) g. You measure its volume in a graduated cylinder using water displacement, and you find the volume to be (7.2 ± 0.1) mL. Calculate the density, including its uncertainty.

A: 1. Calculate the density.

$$\rho = \frac{m}{V} = \frac{24.75\text{ g}}{7.2\text{ mL}} = 3.4375\,\tfrac{\text{g}}{\text{mL}}$$

2. Calculate the relative errors of your two measurements:

$$R.E._{mass} = \frac{\text{uncertainty}}{\text{measured value}} = \frac{0.02}{24.75} = 0.000\,808$$

$$R.E._{volume} = \frac{0.1}{7.2} = 0.013\,889$$

3. Add the individual relative errors together to get the total R.E.:

$$R.E._{mass} + R.E._{volume} = R.E._{total}$$
$$0.000\,808 + 0.013\,889 = 0.014\,697$$

4. Multiply the total R.E. by the density to get the uncertainty:

$$3.4375 \times 0.014697 = 0.050521$$

Because you only estimated one decimal place of uncertainty, you need to round the uncertainty off to ± 0.05.

Because uncertainty is rounded to the hundredths place, you need to also round your answer to the hundredths place:

$$\rho = (3.44 \pm 0.05)\,\tfrac{\text{g}}{\text{mL}}$$

Use this space for summary and/or additional notes:

Honors (not AP®)

Homework Problems

Because the answers are provided, you must show sufficient work in order to receive credit.

1. **(M = Must Do)** In a 4 × 100 m relay race, the four runners' times were: (10.52 ± 0.02) s, (10.61 ± 0.01) s, (10.44 ± 0.03) s, and (10.21 ± 0.02) s. What was the team's (total) time for the event, including the uncertainty?

   Answer: (41.78 ± 0.08) s

2. **(M = Must Do)** A baseball pitcher threw a baseball for a distance of (18.44 ± 0.05) m in (0.52 ± 0.02) s.

   a. What was the velocity of the baseball in meters per second? (Divide the distance in meters by the time in seconds.)

   Answer: $35.46\,\frac{\text{m}}{\text{s}}$

   b. What are the relative errors of the distance and time? What is the total relative error?

   Answer: distance: 0.0027; time: 0.0385; total: 0.0412

   c. Calculate the uncertainty of the velocity of the baseball and express your answer as the velocity (from part a above) plus or minus the uncertainty that you just calculated, with correct rounding.

   Answer: $35.46\,\frac{\text{m}}{\text{s}} \pm 1.46\,\frac{\text{m}}{\text{s}}$ which rounds to $35\,\frac{\text{m}}{\text{s}} \pm 1\,\frac{\text{m}}{\text{s}}$

Use this space for summary and/or additional notes: